Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

then

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^3 \cdot (y_1^2 + bcy_2^2 + cay_3^2 + aby_4^2)$$
$$= |\,1345\,|^2 + bc\,|\,1245\,|^2 + ca\,|\,1235\,|^2 + ab\,|\,1234\,|^2.$$

4. From Lagrange's result it of course follows that

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)(y_1^2 + bcy_2^2 + \cdots)(z_1^2 + bcz_2^2 + \cdots)$$

can be expressed in the same form as any one of the factors, say in the form

$$T_1^2 + bcT_2^2 + caT_3^2 + abT_4^2,$$

and there thus arises the problem of finding the T's in terms of the x's, y's, z's and a, b, c. The simplest way of writing the four expressions obtained is by means of the notation for bipartite functions, according to which

$$\begin{array}{ccc|c} x & y & z & \\ \hline a & b & c & \xi \\ d & e & f & \eta \\ g & h & i & \zeta \end{array} \quad \text{stands for } ax\xi + by\xi + cz\xi + dx\eta + ey\eta + fz\eta + gx\zeta + hy\zeta + iz\zeta,$$

each element of the square array being taken along with the outside element in the same column and at the same time along with the outside element in the same row. We then have

$$T_1 = \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline y_1 & bcy_2 & cay_4 & aby_4 & z_1 \\ -y_2 & y_1 & -ay_4 & ay_3 & bcz_2 \\ -y_3 & by_4 & y_1 & -by_2 & caz_3 \\ -y_4 & -cy_3 & cy_2 & y_1 & abz_4 \end{array}$$

and the other T's differing from it merely in having for their outside column a different row of the determinant equivalent for $(z_1^2 + bcz_2^2 + caz_3^2 + abz_4^2)^2$.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

DISCUSSIONS.

We believe that those of our readers who have followed the discussions of certain solutions of cubic equations as published in the February and June numbers of the Monthly will be especially interested in the method given below for approximating nearly equal roots of a cubic.

I. Relating to Approximations to Nearly Equal Roots of a Cubic Equation.

By Paul Capron, U. S. Naval Academy, Annapolis.

If $y = f(x) \equiv x^3 - 3a^2x + 2ma^3$, the equation $y = 0$ has three real roots when $m^2 \not> 1$, has a double root $\pm a$ when $m = \pm 1$, and has two roots nearly equal to $\pm a$ when m is nearly equal to ± 1. The discussion that follows is confined to the case in which the nearly equal roots are positive; it is applicable to negative roots if the signs of both m and a are reversed, or if the equation is replaced by $\bar{y} \equiv -f(-x) = 0$.

The method of arriving at the approximation is as follows:

Let the larger of the two roots be x_1, the smaller x_2, and let the graph of $f(x)$ be a curve s of which the points P_1, P_2 correspond to the roots x_1, x_2. As the roots are nearly equal, P_1 and P_2 are joined by a relatively short segment of s, which includes the minimum point, L, and does not include the inflection. (These conditions define the restriction implied by the words "nearly equal.") Suppose a quadratic parabola s', having the equation $y_1 = f_1(x)$, to intersect the cubic parabola s three times at L, three times at infinity in the direction of the y axis, and consequently nowhere else. Then s' crosses s at L, and as it is not as steep as s for large values of x, keeps to the right of s where $x > a$, to the left where $x < a$. (At L, $x = a$.)

Let the roots determined by $y_1 = 0$ be x_1', x_2', of which x_1' is the larger. Then $x_1' > x_1$, $x_2' > x_2$. P_1' and P_2' being the points of s' corresponding to the roots x_1', x_2', let ordinates to the cubic s be drawn through P_1' and P_2', meeting s at Q_1 and Q_2. Now let tangents to s at Q_1 and Q_2 meet the x-axis at P_1'' and P_2'', of which the abscissas are x_1'' and x_2''. As the curvature of s does not change sign from P_1 to P_2, it is evident that $x_1'' > x_1$, $x_2'' < x_2$ and also that $f(a) < 0, f(x_1') > 0, f(x_1'') > 0, f(x_2') < 0, f(x_2'') > 0$.

The values $(x_1' - x_1'')$ and $(x_2' - x_2'')$ to be subtracted from x_1' and x_2', are the quotients of the ordinates of s at Q_1 and Q_2 by the slopes of s at these points; it will be found convenient to use a smaller divisor than the actual slope at Q_1, and a (numerically) larger one at Q_2, with the result that, the corrected roots thus found being x_1''' and x_2''', $f(x_1''') < 0$ and $f(x_2''') < 0$; that is, P_1''' and P_2''' being the points of the x-axis having the abscissas x_1''' and x_2''', P_1''' is enough to the left of P_1'' to be to the left of P_1 and P_2''' is enough to the right of P_2'' to be to the right of P_2.

Thus, of the successive approximations to x_1, viz., a, x_1', x_1'', x_1''', the first and last are too small, the others too large. In each case, x''' is the more convenient of the closer approximations, and the error in using it is less than $|x'' - x'''|$. In case the error is undesirably large, a closer approximation, but without limits of error, is obtained by finding the ratio (practically the same in all cases) in which the segment $P''P'''$ is divided by the chord joining the points of s that have the abscissas x'' and x'''.

The resulting approximation may be formulated:

If in the equation $x^3 - 3a^2x + 2ma^3 = 0$, m is nearly equal to 1, let $\frac{1}{9}(1 - m) = \mu$, $\sqrt{\frac{2}{3}(1 - m)} = \lambda$; then the nearly equal roots are

$$x = a(1 \pm \lambda - \mu)$$

with an error less than

$$a\mu \frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda},$$

the larger approximation being too small, the smaller too large. If this limit of error is unsatisfactory, much closer results may be obtained by applying five-sixths of its value as a correction to increase the larger approximation or decrease the smaller approximation.

As an illustration of the method, consider the equation

$$8x^3 - 36x^2 + 39x + 3 = 0.$$

When the roots are diminished by 3/2, this equation is transformed into $x^3 - \frac{15}{8}x + \frac{15}{16} = 0$. Here $a = \frac{1}{4}\sqrt{10}$, $m = \frac{3}{10}\sqrt{10}$, $\lambda = \sqrt{\frac{2}{3}(1 - m)} = 0.184962$, $\mu = \frac{1}{9}(1 - m) = 0.005702$;

$$x = \frac{\sqrt{10}}{4}(1 \pm 0.184962 - 0.005702) = 0.93229 \text{ or } 0.63984.$$

Adding 1.5, we have 2.43229 (too small) and 2.13984 (too large).

The errors are $< a\mu \cdot \dfrac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda} = \dfrac{\sqrt{10}}{4} \times 0.0057 \times \dfrac{0.0925}{1 \pm 0.0925} < 0.0005$.

If the results are not as close as desired, we may compute the errors more carefully (they are 0.00038 + for the larger root, 0.00046 − for the smaller), and, increasing the larger root by $\frac{5}{6} \times 0.00038 = 0.00032$, and decreasing the smaller by $\frac{5}{6} \times 0.00046 = 0.00038$, we may say that the roots are probably 2.43261 and 2.13946. (To six decimals the roots are actually 2.432617 and 2.139465.)

In Wentworth's College Algebra, the equation $x^3 - 515x^2 + 1155x - 649 = 0$ is shown to have two very nearly equal roots, approximately 1.1230914 and 1.1270002. If we multiply the roots of this equation by 3 and diminish the roots of the transformed equation by 515 we have

$$x^3 - 785280x - 267845848 = 0.$$

Here

$$a = 8\sqrt{4090}, \quad m = -66961462\sqrt{4090} \div (65440)^2.$$

$$\sqrt{4090} = 64(1 - p),$$

where

$$p = 0.00073269029253 23856,$$

$$(1 + m) = (65440)^{-2}(-3139968 + 4285533568p) = (65440)^{-2}(0.8435952782),$$

$$(1 + m) = 1.969762360 \times 10^{-10}; \quad \mu = \tfrac{1}{9}(1 + m) = 2.18862384 \times 10^{-11},$$

$$\lambda = \sqrt{\tfrac{2}{3}(1+m)} = (65440)^{-1}\sqrt{0.5623968521}$$
$$= 0.7499312316 \div 65440 = 1.145982933 \times 10^{-5},$$
$$x = -a(1 \pm \lambda - \mu) = -512(1-p)(1 \pm \lambda - \mu)$$
$$= -512 + 512p + 512(1-p)(\mp\lambda + \mu),$$
$$x + 515 = 3 + 0.3751374297765814 + 511.6248625702234186(\mp\lambda + \mu),$$
$$\tfrac{1}{3}(x+515) = 1.123091428324 \text{ or } 1.127000184062.$$

The error is less than

$$\tfrac{1}{3}a\mu\frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda} < \tfrac{1}{3} \times 512(1-p) \times 2.19 \times 10^{-11} \times \frac{5.73 \times 10^{-6}}{1 \pm 5.73 \times 10^{-6}} < 2.14 \times 10^{-14}.$$

In this case, the approximation might safely have been carried one or two places further.

The derivation of the formula follows:

Given $y \equiv f(x) \equiv (x^3 - 3a^2x + 2ma^3)$; transforming the origin of coördinates to the minimum point, L, $(a,\ 2(m-1)a^3)$, we obtain $y_1 = x_1{}^3 \pm 3ax_1{}^2$. The desired quadratic approximation to y_1 near L is $y_2 = 3ax_1{}^2$. Transformed to the original system of coördinates, the latter becomes $y_3 = 3ax^2 - 6a^2x + (1+2m)a^3$. Solving $y_3 = 0$ we find $x_1' = a(1 + \sqrt{\tfrac{2}{3}(1-m)})$, $x_2' = a(1 - \sqrt{\tfrac{2}{3}(1-m)})$.

Substituting in $y = f(x) \equiv x^3 - 3a^2x + 2ma^3$, we obtain $f(x_1') = a^3[\tfrac{2}{3}(1-m)]^{3/2}$, $f(x_2') = -a^3[\tfrac{2}{3}(1-m)^{3/2}]$.

The derivative is $f'(x) \equiv 3(a^2 - x^2)$; $f'(x_1') = 2a^2(1 - m + \sqrt{6(1-m)}) > 0$, $f'(x_2') = 2a^2(1 - m - \sqrt{6(1-m)}) < 0$.

$$x_1' - x_1'' = \frac{f(x_1')}{f'(x_1')} = a\frac{[\frac{2}{3}(1-m)]^{3/2}}{2[1 - m + 3\sqrt{\frac{2}{3}(1-m)}]} = \frac{a}{9}\cdot\frac{1-m}{1 + \frac{1}{2}\sqrt{\frac{2}{3}(1-m)}};$$

similarly

$$x_2' - x_2'' = \frac{a}{9}\cdot\frac{1-m}{1 - \frac{1}{2}\sqrt{\frac{2}{3}(1-m)}},$$

$$x_1'' = a\left[1 + \sqrt{\tfrac{2}{3}(1-m)} - \frac{1}{9}\frac{1-m}{1 + \frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}\right];$$

$$x_2'' = a\left[1 - \sqrt{\tfrac{2}{3}(1-m)} - \frac{1}{9}\cdot\frac{1-m}{1 - \frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}\right].$$

Neglecting the radical in the denominator, we have:

$$x_1''' = a[1 + \sqrt{\tfrac{2}{3}(1-m)} - \tfrac{1}{9}(1-m)] < x_1'';$$

$$x_2''' = a[1 - \sqrt{\tfrac{2}{3}(1-m)} - \tfrac{1}{9}(1-m)] > x_2''.$$

Also

$$f(x_1''') = \frac{a^3}{27}(1-m)^2[-5+\sqrt{\tfrac{2}{3}(1-m)}-\tfrac{1}{27}(1-m)] < 0.$$

$$f(x_2''') = \frac{a^3}{27}(1-m)^2[-5-\sqrt{\tfrac{2}{3}(1-m)}-\tfrac{1}{27}(1-m)] < 0.$$

Therefore the errors are less, respectively, than

$$x_1'' - x_1''' = \frac{a}{9}(1-m)\left[1-\frac{1}{1+\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}\right] = \frac{a}{9}(1-m)\cdot\frac{\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}{1+\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}},$$

$$x_2''' - x_2'' = \frac{a}{9}(1-m)\left[\frac{1}{1-\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}-1\right] = \frac{a}{9}(1-m)\cdot\frac{\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}{1-\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}.$$

Hence the rule given earlier for nearly equal positive roots. For nearly equal negative roots (m nearly -1), if $\frac{1}{9}(1+m) = \mu$, $\sqrt{\frac{2}{3}(1+m)} = \lambda$, $x = -a(1 \pm \lambda - \mu)$, with an error less than $a\mu\cdot(\frac{1}{2}\lambda/(1 \pm \frac{1}{2}\lambda))$, the numerically larger root being too small numerically, the other too large numerically.

Whether the approximate roots are positive or negative, the one further from the inflection is too small, the one nearer the inflection too large; in other words, both are on the portion of the x-axis intercepted by the adjacent arch of the graph of $y = 0$.

If we let $\frac{1}{2}\mu = z$, we have $f(x_1''') = -12a^3z^2(5-2\sqrt{3z}-6z)$ as already seen, and further, $f(x_1'') = 12a^3z^2(1-3z)^{-2}(1-\frac{38}{3}z+21z^2+8z(1-2z)\sqrt{3z})$; $f(x_2''')$ and $f(x_2'')$ differ only in the sign of $\sqrt{3z}$. Consequently

$$-f(x_1''') : f(x_1'') = 5 : 1$$

very nearly, for either root. Approximating by means of the chord of the graph, we thus obtain a closer result, for which, however, no limits of error are fixed. If the limit of error of the earlier approximation is

$$a\mu\frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda} = E_1,\, E_2$$

(E_1 taking the upper sign, E_2 the lower)

$$x = a(1 \pm \lambda - \mu \pm \tfrac{5}{6}E_{1,\,2})$$

if the roots are positive;

$$x = -a(1 \pm \lambda - \mu \pm \tfrac{5}{6}E_{1,\,2})$$

if the roots are negative.

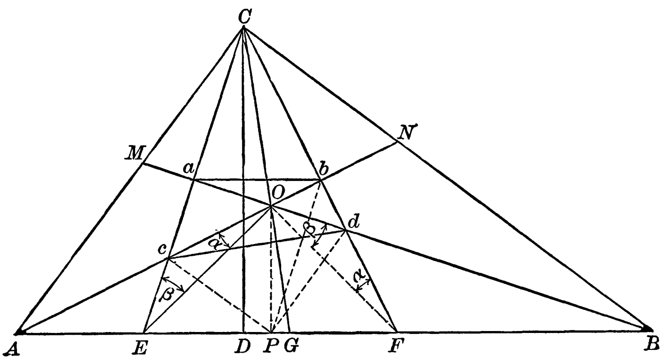
The chief advantages of the approximation are that it gives an easy means of locating two roots, and of separating them if they are very nearly equal, and in the latter case gives an approximate solution more convenient than Horner's method and more accurate than the trigonometric solution. The formulas for

the trigonometric solution are $\cos 3\theta = -m$, $x = 2a \cos \theta$; if $|m|$ is nearly unity, the value of θ is not very accurately determined. The first of the examples cited above is near the border-line; five-place tables will give nearly as accurate results by the trigonometric method as by the approximation, but with more labor. [The roots by this method are -0.0721, 2.4326, 2.1394.]

II. Relating to Some Relations in a Right-angled Triangle.

By Albert Babbitt, University of Nebraska, Lincoln.

In the right-angled triangle ABC (with right angle at C) draw CD perpendicular to AB; also draw CE, CF, BM and AN bisectors of the angles ACD, BCD, ABC and CAB respectively.



We have then the following theorems:

Theorem I. *The bisector of the angle B is perpendicular to CE, and the bisector of the angle A is perpendicular to CF. Moreover, CE is bisected by the bisector of the angle B (in point a) and CF is bisected by the bisector of the angle A (in pt. b), and consequently ab is parallel to AB and is equal to $\frac{1}{2}EF$.*

Proof. Since

$$\angle bOd = \frac{A}{2} + \frac{B}{2} = 45^\circ,$$

$$\angle bdO = \frac{BCD}{2} + \frac{B}{2} = 45^\circ,$$

hence $\angle dbO = 90^\circ$, and AN is perpendicular to CF. Similarly, it may be proved that BM is perpendicular to CE.

From the equality of the right-angled triangles FbA and AbC, we have $Fb = bC$, i. e., CF is bisected by the bisector of the angle A. Similarly, $Ea = aC$, and it follows at once that ab is parallel to AB and is equal to $\frac{1}{2}EF$.

Theorem II. *The bisectors CF and CE cut off on the hypotenuse AB segments AF and BE which are equal to the sides AC and BC respectively.*

Proof. From the equality of the right triangles FbA and AbC, it follows that